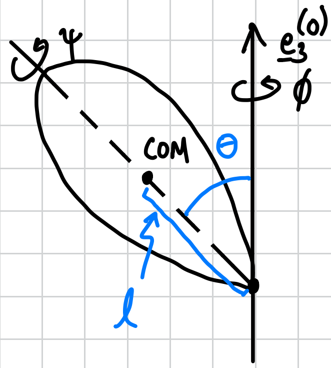
Lecture 13

The Lagrangian is



$$\boxed{\mathcal{L}(\phi,\theta,\psi,\dot\phi,\dot\theta,\dot\psi) = \underbrace{\frac{1}{2}\left(\overset{I_{33}}{C}(\dot\psi+\dot\phi\cos\theta)^2 + \overset{I_{11}=I_{22}}{A}(\dot\theta^2+\dot\phi^2\sin^2\theta)\right)}_{K.E} - \underbrace{Mg\ell\cos\theta}_{\text{potential}}}$$

<u>Remarks:</u>

(i) $\mathcal{L}$ does not depend on ψ

$$\frac{\partial \mathcal{L}}{\partial \psi} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\psi}\right) = \frac{d}{dt} p_\psi = 0$$

$$\Rightarrow p_\psi = a \text{ is constant}$$

But $\boxed{p_\psi = a = C(\dot\psi + \dot\phi\cos\theta)} = \text{constant}$

(ii) $\mathcal{L}$ does not depend on ϕ

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\phi}\right) = \frac{d}{dt} p_\phi = 0$$

$$\Rightarrow p_\phi = b \text{ is constant}$$

But $\boxed{p_\phi = b = A\dot\phi\sin^2\theta + C(\dot\psi + \dot\phi\cos\theta)\cos\theta} = \text{constant}$

(iii) $\mathcal{L}$ does not depend on t, so

$$\frac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow \dot\phi p_\phi + \dot\theta p_\theta + \dot\psi p_\psi - \mathcal{L} \text{ is conserved}$$

$$\Rightarrow \frac{1}{2}A(\dot\theta + \dot\phi\sin^2\theta) + \frac{1}{2}C(\dot\psi + \dot\phi\cos\theta)^2 + Mg\ell\cos\theta \equiv E \text{ is conserved}$$

↳ an equation for θ ; $\dot\theta^2 = \ldots\ldots$

$$\Rightarrow \boxed{E = \frac{1}{2}A(\dot\theta^2 + \dot\phi^2\sin^2\theta) + \frac{1}{2}C\underbrace{(\dot\psi + \dot\phi\cos\theta)^2}_{\frac{a^2}{C^2}} + Mg\ell\cos\theta}$$

Initial conditions will provide a, b and E and really only needs θ to be determined because

$$\dot{\phi} = \frac{b - a\cos\theta}{A\sin^2\theta} \quad , \quad \dot{\psi} = \frac{a}{C} - \dot{\phi}\cos\theta = \frac{a}{C} - \left(\frac{b - a\cos\theta}{A\sin^2\theta}\right)\cos\theta$$

Note: $E = \frac{1}{2}A\dot{\theta}^2 + \frac{1}{2}\frac{(b - a\cos\theta)^2}{A\sin^2\theta} + \frac{a^2}{2C} + Mgl\cos\theta$

Define $\quad \hat{E} \equiv E - \frac{a^2}{2C} \quad \Rightarrow \quad \boxed{\hat{E} = E - \frac{a^2}{2C} = \frac{1}{2}A\dot{\theta}^2 + \frac{(b - a\cos\theta)^2}{2A\sin^2\theta} + Mgl\cos\theta}$

(rearranging) $\quad \Rightarrow \quad \boxed{\dot{\theta}^2 = \frac{2\hat{E}}{A} - \frac{1}{A^2}\frac{(b - a\cos\theta)^2}{\sin^2\theta} - \frac{2Mgl}{A}\cos\theta \equiv F(\theta)}$

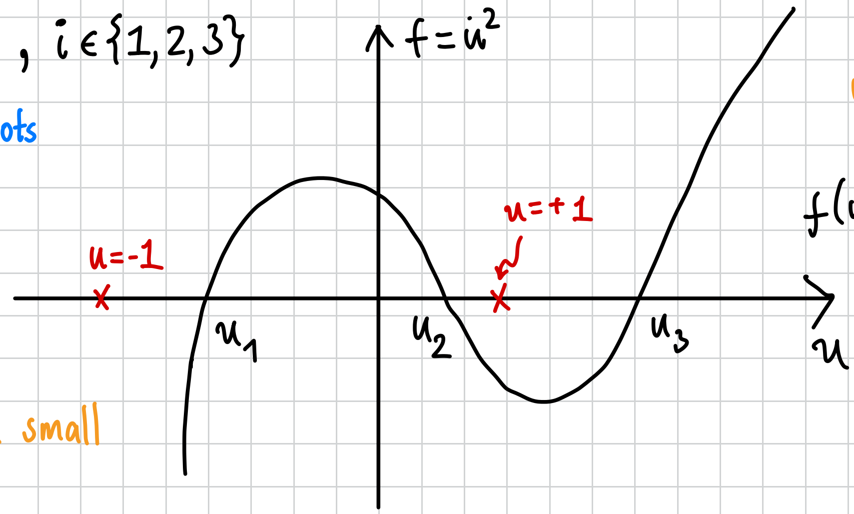
Let $u = \cos\theta \Rightarrow \dot{u} = -\dot{\theta}\sin\theta$

$$\Rightarrow \dot{\theta}^2 = \frac{\dot{u}^2}{\sin^2\theta} = \frac{2\hat{E}}{A} - \frac{1}{A^2}\frac{(b - au)^2}{(1 - u^2)} - \frac{2Mgl}{A}u$$

$$\therefore \boxed{f(u) = \dot{u}^2 = \frac{2\hat{E}}{A}(1 - u^2) - \frac{1}{A^2}(b - au)^2 - \frac{2Mgl}{A}u(1 - u^2)} = \text{cubic function of } u$$

$f(u_i) = 0, \; i \in \{1, 2, 3\}$

↳ roots



$f(u_i)$ depend on $a, b, \hat{E}$

Since left hand of $f(u) = \dot{u}^2 \geq 0$, u constrained to intevals for which the cubic polynomial $f(u)$ is positive on RHS

Also if $u = \pm 1$, $f(u) = -\frac{1}{A}(b - a)^2 \leq 0$, hence $u = \pm 1$ on interval where $f(u)$ is $-$ve

Note $\quad u = \cos\theta \Rightarrow -1 \leq u \leq 1$

Both $u = 1, -1$ typically such that $u_2 < 1 < u_3$, $-1 < u_1$ because $f(\pm 1) < 0$

Typically motion constrained so that $u_1 \leq u \leq u_2$

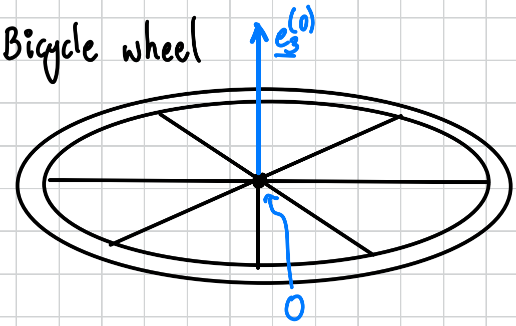
(iv) Equation of motion for θ (Euler-Lagrange equation for θ):

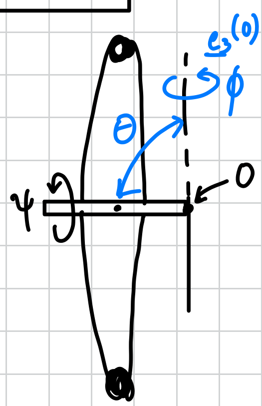
$$0 = \frac{\partial \mathcal{L}}{\partial \theta} - \underbrace{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta}\right)}_{A\ddot\theta} \Rightarrow A\ddot\theta = \frac{\partial \mathcal{L}}{\partial \theta}$$

Hence
$$A\ddot\theta = A\sin\theta\cos\theta\,\dot\phi^2 - \overbrace{C\sin\theta(\dot\psi + \cos\theta\,\dot\phi)\dot\phi}^{a\dot\phi\sin\theta} + Mgl\sin\theta$$

$$\Rightarrow \boxed{A\ddot\theta = A\dot\phi^2\sin\theta\cos\theta - a\dot\phi\sin\theta + Mgl\sin\theta}$$

Example: Bicycle wheel





steady solution with $\theta = \frac{\pi}{2}$

So at $\theta = \frac{\pi}{2}$, $\ddot\theta = 0 \Rightarrow Mgl = a\dot\phi$

$$\Rightarrow \dot\phi = \frac{Mgl}{a} \quad \text{(precession)}, \quad \dot\psi = \frac{a}{C} \leftarrow \text{set this value}$$

To test stability, we perturb it.

Lets put $\theta = \frac{\pi}{2} + \varepsilon$ ← small value

Therefore, we get

$$\sin\theta = \sin\left(\frac{\pi}{2} + \varepsilon\right) = \cos\varepsilon \approx 1 - \frac{\varepsilon^2}{2} \leftarrow \varepsilon \text{ is small}$$

$$\cos\theta = \cos\left(\frac{\pi}{2} + \varepsilon\right) = -\sin\varepsilon \simeq \varepsilon$$

Eqn of motion becomes ← ignore all second order and higher approximation

$$A\ddot\theta = A\ddot\varepsilon \simeq -A\dot\phi^2\varepsilon + O(\varepsilon^2, \ldots) \leftarrow \text{big O notation}$$
$$= -A\dot\phi^2\varepsilon + O(\varepsilon^2, \ldots)$$

$$\Rightarrow \boxed{\ddot\varepsilon = -\dot\phi^2\varepsilon}$$

↳ periodic soln

$\Rightarrow$ solution is stable

Example 2

Initial conditions:

$$\theta(0) = \frac{\pi}{2}, \quad \dot\psi(0) = n, \quad \dot\theta(0) = \dot\phi(0) = \frac{Cn}{A}$$

Then

$$a = C(\dot\psi + \dot\phi\cos\theta)\Big|_{t=0} = Cn$$

$$b = (A\dot\phi\sin^2\theta + a\cos\theta)_{t=0} = Cn$$

$$\hat{E} = (\ldots)\Big|_{t=0} = \frac{C^2n^2}{A}$$

The cubic polynomial

$$\dot u^2 = f(u) = \frac{2}{A}\left(\frac{C^2n^2}{A} - Mglu\right)(1-u^2) - \frac{1}{A^2}C^2n^2(1-u)^2 \qquad (a=b)$$

$$= \frac{2C^2n^2}{A^2}\left[\left(1 - \underbrace{\frac{MglA}{C^2n^2}}_{\beta>0} u\right)(1+u) - \frac{1}{2}(1-u)\right](1-u)$$

$$= \frac{2C^2n^2}{A^2}(1-u)\left(\frac{1}{2} + \left(\frac{3}{2} - \beta\right) - \beta u^2\right)$$